AP STATISTICS
TOPIC IV: FINITE PROBABILITY SPACES

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1. Probability Spaces

1.1. **Probability Spaces.** A study of probability is on firm ground when it uses the concepts of sets and functions to precisely define its terms. Thus measurement of probabilities takes place in a formal mathematical object known as a probability space.

Definition 1. A *finite probability space* consists of a finite set S together with a function

$$p : S \to [0,1] \quad \text{satisfying} \quad \sum_{s \in S} p(s) = 1.$$

Let $\mathcal{E}$ denote the set of all subsets of S. Then p determines a function

$$P : \mathcal{E} \to [0,1] \quad \text{given by} \quad P(E) = \sum_{s \in E} p(s).$$

The set S is called the *sample space*. The elements of S are called *outcomes*. The members of $\mathcal{E}$ are called *events*. The function P is called a *probability measure*. The number $P(E)$ is called *probability* of event E.

Proposition 1. *Let S be a finite probability space. Let $A, B \subset S$. Then*

- **(a)** $P(\varnothing) = 0$;
- **(b)** $P(A^c) = 1 - P(A)$;
- **(c)** $A \subset B \Rightarrow P(A) \leq P(B)$;
- **(d)** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$;

Corollary 1. (Boole's Inequality)
Let S be a finite probability space. Let $A, B \subset S$. Then

$$P(A \cup B) \leq P(A) + P(B).$$

Definition 2. Let S be a finite set. The *uniform probability space* on S defined by the function

$$p : S \to [0,1] \quad \text{given by} \quad p(s) = \frac{1}{|S|}.$$

Then the probability measure on the collection $\mathcal{E}$ of all subsets of S is

$$P : \mathcal{E} \to [0,1] \quad \text{given by} \quad P(E) = \frac{|E|}{|S|}.$$

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1.2. **Examples of Probability Spaces.** We begin with examples of uniform probability spaces.

Example 1. Let $S = \{\mathtt{T}, \mathtt{H}\}$. This models flipping a coin, and $p(s) = \dfrac{1}{2}$ for $s \in S$.

Example 2. Let $S = R \times U$, where $R = \{\mathtt{2}, \mathtt{3}, \mathtt{4}, \mathtt{5}, \mathtt{6}, \mathtt{7}, \mathtt{8}, \mathtt{9}, \mathtt{10}, \mathtt{J}, \mathtt{Q}, \mathtt{K}, \mathtt{A}\}$ is the set of ranks and $U = \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ is the set of suits. This corresponds to drawing one card from a deck of 52 cards, with $p(s) = \dfrac{1}{52}$ for each $s \in S$.

Example 3. Let $S = \{1, 2, 3, 4, 5, 6\}$. This models rolling a fair die, and $p(s) = \dfrac{1}{6}$ for $s \in S$.

Example 4. Let $D = \{1, 2, 3, 4, 5, 6\}$ represents rolling one die; then $S = D \times D$ represents rolling a pair of distinguishable dice (say they are of two different colors). Then $p(s) = \dfrac{1}{36}$ for $s \in S$. Let E the the subset of S consisting of ordered pairs whose sum is 5. Then $E = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$, and $P(E) = \frac{|E|}{|S|} = \dfrac{4}{36} = \dfrac{1}{9}$.

Now we give a natural example of a nonuniform probability space.

Example 5. Let $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, and for $s \in S$, let $p(s)$ equal the probability of rolling two dice whose sum is s. Then $p(2) = \frac{1}{36}$, $p(3) = \frac{1}{18}$, $p(4) = \frac{1}{12}$, $p(5) = \dfrac{1}{9}$, and so forth. This is clearly not uniform.

Example 6. Ten ping pong balls numbered 1 through 10 are in a bin. Three are drawn at random. We win if all of the numbers on them are odd. Find an appropriate sample space S and event E to model this problem, and compute $P(E)$.

Solution. Let $B = \{1, 2, \ldots, 10\}$; we take this to be the set of balls. This is NOT the sample space.

Note the order in which the balls are drawn does not matter in this experiment, and the balls are not replaced as we select them. so an outcome is a subset of B containing three distinct numbers. The sample space is the set of all subsets of B of cardinality 3:

$$S = \{A \subset B \mid |A| = 3\}.$$

We obtain and outcome by picking three balls in any order, so the number of outcomes is 10 choose 3:

$$|S| = \binom{10}{3} = 120.$$

The event is the set of a subsets of B which contain three odd balls. Set

$$E = \{A \in S \mid n \in A \Rightarrow n \text{ is odd }\}.$$

An outcome from E is obtained by selecting 3 odd numbers from B, and there are 5 choose 3 ways to do this:

$$|E| = \binom{5}{3} = 10.$$

Thus

$$P(E) = \frac{\binom{5}{3}}{\binom{10}{3}} = \frac{10}{120} = \frac{1}{12} \approx 8.33\%.$$

$\square$

2. Compound Events

We now investigate what happens when two or more events are under consideration. They can interact in various ways.

2.1. Disjointness.

We consider the conditions under which the probability of either of two events occurring is the sum of the probabilities of the events.

Definition 3. Let S be a finite probability space. Let $A, B \subset S$.
We say that A and B are *disjoint* (or *mutually exclusive*) if

$$A \cap B = \varnothing.$$

Proposition 2. *Let S be a finite probability space. Let $A, B \subset S$ be disjoint. Then*

$$P(A \cup B) = P(A) + P(B).$$

Proposition 3. *Let S be a finite probability space. Let $\{A_1, \dots, A_n\}$ be a partition of S. Let $E \subset S$. Then*

$$P(E) = \sum_{i=1}^{n} P(E \cap A_i).$$

The following is called the *inclusion-exclusion principle*, and can be generalized to any number of sets.

Proposition 4. (Inclusion-Exclusion Principle)
Let S be finite probability space, with $E \subset S$. Suppose that $E = A \cup B \cup C$. Then

$$P(E) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

This can be rephrased as

$$P(E) = P(A) + P(B \cap A^c) + P(C \cap A^c \cap B^c).$$

2.2. Independence.

We consider the conditions under which the probability of both of two events occurring is the product of the probabilities of the events.

Definition 4. Let S be a finite probability space. Let $A, B \subset S$.
We say that A and B are *independent* if

$$P(A \cap B) = P(A)P(B).$$

Example 7. Let S be a set of 52 cards. Let A be the set of spades and let B be the set of aces. Then

$$P(A \cap B) = \frac{1}{52} = \frac{1}{4}\frac{1}{13} = P(A)P(B).$$

Thus A and B are independent events.

2.3. **Conditioning.** We consider the computation of the probability of an event occurring given that some other event occurred.

Definition 5. Let S be a finite probability space. Let $A, B \subset S$ with $P(B) > 0$. Define the *conditional probability* of A with respect to B to be

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Glorious Conditioning Theorem 1. (Multiplication Rule)
Let S be a finite probability space and let $A, B \subset S$ with $P(B) > 0$. Then

$$P(A \cap B) = P(A|B)P(B).$$

Glorious Conditioning Theorem 2. (Total Probabilities Rule)
Let S be a finite probability space and let $A \subset S$. Let $\{B_1, \ldots, B_n\}$ be a partition of S, with $P(B_i) > 0$ for $i = 1, \ldots, n$. Then

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i).$$

Glorious Conditioning Theorem 3. (Bayes Rule)
Let S be a finite probability space and let $A \subset S$. Let $\{B_1, \ldots, B_n\}$ be a partition of S, with $P(B_i) > 0$ for $i = 1, \ldots, n$. Then

$$\begin{aligned} P(B_j|A) &= \frac{A \cap B_j}{A} \\ &= \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}. \end{aligned}$$

3. Problems

Problem 1. Romeo and Juliet attend Capulet's ball. They are among 10 boys and 10 girls who are randomly paired for a dance, one boy with one girl. How likely is it that Romeo is paired with Juliet?

(a) Find the sample space, and find the cardinality of the sample space.
(b) Find the event, and find the cardinality of the event.
(c) Find the probability of the event.

Problem 2. Bob is a lobster in a tank with six other lobsters. The chef reaches in, randomly pulls out two lobsters, and throws them into a pot of boiling water, where they perish in agony. What are the chances that Bob survives?

(a) Find the sample space, and find the cardinality of the sample space.
(b) Find the event, and find the cardinality of the event.
(c) Find the probability of the event.

Problem 3. A fair die is rolled three times. Sam bets that three consecutive increasing numbers will be rolled. What are Sam's chances of winning?

(a) Find the sample space, and find the cardinality of the sample space.
(b) Find the event, and find the cardinality of the event.
(c) Find the probability of the event.

Problem 4. A parking lot contains twelve parking spaces in a row. Seven cars park in randomly selected spaces. What are the chances that there are six consecutive vacant spaces?

(a) Find the sample space, and find the cardinality of the sample space.
(b) Find the event, and find the cardinality of the event.
(c) Find the probability of the event.

Problem 5. A coin is flipped 5 times. What is the likelihood of getting 3 heads?

(a) Find the sample space, and find the cardinality of the sample space.
(b) Find the event, and find the cardinality of the event.
(c) Find the probability of the event.

Problem 6. A bin contains 5 red balls, 7 blue balls, and 11 white balls. Ten balls are drawn at random. What are the chances that all of the blue balls, and none of the red balls, are drawn?

(a) Find the sample space, and find the cardinality of the sample space.
(b) Find the event, and find the cardinality of the event.
(c) Find the probability of the event.

Problem 7. Five cards are dealt from a shuffled deck. What are the chances that at least three of them are from the same suit?

(a) Find the sample space, and find the cardinality of the sample space.
(b) Find the event, and find the cardinality of the event.
(c) Find the probability of the event.

Problem 8. Five cards are dealt from a shuffled deck and placed in a line. What is the likelihood that they are in consecutive increasing order?

(a) Find the sample space, and find the cardinality of the sample space.
(b) Find the event, and find the cardinality of the event.
(c) Find the probability of the event.

Problem 9. Twenty balls are thrown at random into five bins. What is the likelihood that at least one bin is empty?

(a) Find the sample space, and find the cardinality of the sample space.
(b) Find the event, and find the cardinality of the event.
(c) Find the probability of the event.

Problem 10. A coin is flipped ten times. Determine the sample space, and find the probability of each event.

(a) All values are tails.
(b) Exactly 7 values were heads.
(c) At least 7 values were heads.

Problem 11. A bin contains 40 red balls and 20 blue balls. Ten balls are drawn at random. Determine the sample space, and find the probability of each event.

(a) All balls are red.
(b) Exactly 7 balls are red.
(c) At least 7 balls are red.

Problem 12. Four cards are dealt from a shuffled deck and placed in a line. Determine the sample space, and find the probability of each event.

(a) The cards are in increasing order.
(b) The cards all have the same suit.
(c) The cards all have the same rank.

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